MATH4240: Stochastic Processes Tutorial 3

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Recurrent/ Transient

Denote by

$$\rho_{xy} = P_x(T_y < \infty)$$

the probability that the chain from x returns back to y in finite time. If $\rho_{xx} = 1$, we call x a *recurrent* state. Otherwise, we call x a *transient* state.

Two-state Markov chain

Let $\{X_n\}_{n\geq 0}$ be the two-state Markov chain (page 2 in textbook) with the state space $\mathcal{S} = \{0, 1\}$ and the transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix},$$

where $0 < p, q < 1$.
Find ρ_{00}.

Two-state Markov chain

$$\begin{aligned}
\rho_{00} &= P_0(T_0 < \infty) \\
&= P(X_0 = X_1 = 0) + P(X_0 = 0, X_1 = 1, X_2 = 0) + \\
&\qquad P(X_0 = 0, X_1 = X_2 = 1, X_3 = 0) + \dots \\
&= (1-p) + pq + p(1-q)q + p(1-q)^2 q + \dots \\
&= (1-p) + pq \sum_{k=0}^{\infty} (1-q)^k \\
&= (1-p) + \frac{pq}{1-(1-q)} \\
&= 1
\end{aligned}$$

Similarly. we also have $\rho_{11} = 1$, i.e. every state is recurrent.

One-Step Calculations on hitting probabilities

In the textbook, question 9 on page 42 says: using the formula

$$P_x(T_y = n+1) = \sum_{z \neq y} P(x,z) P_z(T_y = n), \quad n \geq 1 \tag{1}$$

to verify the following identity

$$\rho_{xy} = P(x,y) + \sum_{z \neq y} P(x,z)\rho_{zy}. \tag{2}$$

Proof. By formula (2),

$$
\begin{aligned}
P_x(T_y \leq n+1) &= \sum_{k=0}^{n} P_x(T_y = k+1) \\
&= P_x(T_y = 1) + \sum_{k=1}^{n} P_x(T_y = k+1) \\
&= P(x,y) + \sum_{k=1}^{n} \left(\sum_{z \neq y} P(x,z) P_z(T_y = k) \right) \\
&= P(x,y) + \sum_{z \neq y} P(x,z) \sum_{k=1}^{n} P_z(T_y = k) \\
&= P(x,y) + \sum_{z \neq y} P(x,z) P_z(T_y \leq n), \quad n \geq 0.
\end{aligned}
$$

One-Step Calculations on hitting probabilities

As in the definition $\rho_{xy} = P_x(T_y < \infty)$, we have

$$
\begin{aligned}
\rho_{xy} &= \lim_{n\to\infty} P_x(T_y \le n+1) \\
&= \lim_{n\to\infty} \left(P(x,y) + \sum_{z\neq y} P(x,z) P_z(T_y \le n) \right) \\
&\qquad \text{(by Monotone Convergence Theorem)} \\
&= P(x,y) + \sum_{z\neq y} P(x,z) \left(\lim_{n\to\infty} P_z(T_y \le n) \right) \\
&= P(x,y) + \sum_{z\neq y} P(x,z) P_z(T_y < \infty) \\
&= P(x,y) + \sum_{z\neq y} P(x,z) \rho_{zy}.
\end{aligned}
$$

One-Step Calculations on expected values of hitting times

In a *finite* irreducible Markov chain (in particular $\rho_{xy} = 1$ for all $x, y \in \mathcal{S}$), another important formula can be induced from the formula (2):

$$E_x(T_y) = 1 + \sum_{z \neq y} P(x,z) E_z(T_y). \tag{3}$$

Proof. In a finite irreducible MC, $\rho_{xy} = 1$ for any $x, y \in \mathcal{S}$. By formula (2),

$$\begin{aligned}
E_x(T_y) &= \sum_{n=0}^{\infty}(n+1)P_x(T_y = n+1) \\
&= P_x(T_y = 1) + \sum_{n=1}^{\infty}(n+1)P_x(T_y = n+1) \\
&= P(x,y) + \sum_{n=1}^{\infty}(n+1)\left(\sum_{z \neq y} P(x,z)P_z(T_y = n)\right) \\
&\qquad \text{(since the state space is finite)} \\
&= P(x,y) + \sum_{z \neq y} P(x,z)\left(\sum_{n=1}^{\infty}(n+1)P_z(T_y = n)\right)
\end{aligned}$$

One-Step Calculations on expected values of hitting times

$$
\begin{aligned}
&= P(x,y) + \sum_{z \neq y} P(x,z) \left(\sum_{n=1}^{\infty} n P_z(T_y = n) + \sum_{n=1}^{\infty} P_z(T_y = n) \right) \\
&= P(x,y) + \sum_{z \neq y} P(x,z) \left(E_z(T_y) + P_z(T_y < \infty) \right) \\
&= P(x,y) + \sum_{z \neq y} P(x,z) \left(E_z(T_y) + \rho_{zy} \right) \\
&= P(x,y) + \sum_{z \neq y} P(x,z) \left(E_z(T_y) + 1 \right) \\
&= \sum_{z \in \mathcal{S}} P(x,z) + \sum_{z \neq y} P(x,z) E_z(T_y) \\
&= 1 + \sum_{z \neq y} P(x,z) E_z(T_y).
\end{aligned}
$$

Example: Ehrenfest Chain

Consider the Ehrenfest chain with $d = 3$.

(a) Find $P_x(T_0 = n)$ for $x \in \{0, 1, 2, 3\}$ and $1 \leq n \leq 3$.

(b) Find ρ_{10}, ρ_{20}, and ρ_{30}.

(c) Find $E_3(T_0)$.

Example: Ehrenfest Chain

Solution.

(a) The transition matrix is

	0	1	2	3
0	0	1	0	0
1	1/3	0	2/3	0
2	0	2/3	0	1/3
3	0	0	1	0

For $n = 1$, $P_x(T_0 = 1) = P(x, 0) = \begin{cases} 1/3, & \text{if } x = 1; \\ 0, & \text{otherwise.} \end{cases}$

Example: Ehrenfest Chain

For $n = 2$, by formula (1),

$$\begin{aligned} P_x(T_0 = 2) &= \sum_{y \neq 0} P(x, y) P_y(T_0 = 1) \\ &= \sum_{y \neq 0} P(x, y) P(y, 0) \\ &= P(x, 1) P(1, 0) = \begin{cases} 1/3, & \text{if } x = 0; \\ 2/9, & \text{if } x = 2; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example: Ehrenfest Chain

For $n = 3$, by formula (1),

$$\begin{aligned} P_x(T_0 = 3) &= \sum_{y \neq 0} P(x, y) P_y(T_0 = 2) \\ &= P(x, 2) P_2(T_0 = 2) = \begin{cases} 4/27, & \text{if } x = 1; \\ 2/9, & \text{if } x = 3; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example: Ehrenfest Chain

(b) By formula (2),

$$\begin{cases} \rho_{30} = P(3,0) + \sum\limits_{z \neq 0} P(3,z)\rho_{z0} = \rho_{20}, \\ \rho_{20} = P(2,0) + \sum\limits_{z \neq 0} P(2,z)\rho_{z0} = (2/3)\rho_{10} + (1/3)\rho_{30}, \\ \rho_{10} = P(1,0) + \sum\limits_{z \neq 0} P(1,z)\rho_{z0} = 1/3 + (2/3)\rho_{20}. \end{cases}$$

Hence $\rho_{10} = \rho_{20} = \rho_{30} = 1$.

Example: Ehrenfest Chain

(c) By formula (3),

$$\begin{cases} E_3(T_0) = 1 + \sum_{z \neq 0} P(3,z)E_z(T_0) = 1 + E_2(T_0), \\ E_2(T_0) = 1 + \sum_{z \neq 0} P(2,z)E_z(T_0) = 1 + (2/3)E_1(T_0) + (1/3)E_3(T_0), \\ E_1(T_0) = 1 + \sum_{z \neq 0} P(1,z)E_z(T_0) = 1 + (2/3)E_2(T_0). \end{cases}$$

Hence $E_3(T_0) = 10$.

Example: Duration of Fair Games.

Consider the gambler's ruin chain in which $P(i, i+1) = P(i, i-1) = 1/2$ for $0 < i < N$ and the end points are absorbing states: $P(0,0) = P(N,N) = 1$. Let $T = \min\{T_0, T_N\}$ be the time at which the chain enters an absorbing state. Find $E_n(T)$ for each $n \in \{1, 2, \cdots, N-1\}$.

Example: Duration of Fair Games.

Solution. By formula (3),

$$E_j(T) = \frac{1}{2}E_{j-1}(T) + \frac{1}{2}E_{j+1}(T) + 1, \quad j = 1, \cdots, N-1. \tag{4}$$

Let $h(j) = E_j(T) - E_{j-1}(T)$, $j = 1, \cdots, N$, then by (4),

$$h(j) = h(j+1) + 2.$$

Hence

$$0 = E_N(T) - E_0(T) = \sum_{j=1}^{N} h(j) = \sum_{j=1}^{N} (h(1) - 2(j-1)) = N \cdot h(1) - N(N-1)$$

which implies $h(1) = N - 1$. Therefore, for $n \in \{1, 2, \cdots, N-1\}$,

$$E_n(T) = \sum_{j=1}^{n} h(j) + E_0(T) = \sum_{j=1}^{n} (h(1) - 2(j-1)) = n(N-n).$$